Implications of Griffith's condition on crack growth

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We prove that antiplane crack growth governed by the Griffith's condition and the diffusion-limited-aggregation (DLA) growth law is equivalent to the dielectric-breakdown model with cutoffs on the gradients, introduced by Arian *et al.* [Phys. Rev. Lett. **63**, 2005 (1989)]. We conclude that the crack will evolve from DLA-type patterns (at small sizes) into spiky structures (at large sizes). The same conclusion holds for general crack growth.

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Since the pioneering work of Mandelbrot, Passoja, and Paulley [1] the efforts made to understand fracture as a growth process have been continuously increasing. Many crack-growth models have been introduced and some significant achievements have been attained [1–7]. But we do not have a deep understanding of fracture process.

In [8] Arian *et al.* introduced the dielectric breakdown (DB) models with lower cutoffs, which represent realistic situations in some DB and some viscous fingering experiments. In this Brief Report, we show that the same situation exists in crack growth.

Griffith [9,10] was the first to apply thermodynamics to fracture, to our knowledge. He considered a crack in a brittle material. Based on the law of conservation of energy, he reasoned that during an increment of crack extension da there can be no change in the total energy E composed of the sum of the potential energy of deformation Π and the surface energy S, i.e.,

$$dE = d\Pi + dS = 0 \ . \tag{1}$$

Equation (1) can be written as

$$G = G_c \ , \tag{2}$$

where $G = -d\Pi/da$ is known as the energy release rate and $G_c = dS/da$ is a characteristic of the material. Equation (2) represents the fracture criterion.

Now we consider a two-dimensional linear elastic antiplane system, defined on the lattice, under a fixed-displacement boundary condition. Antiplane crack growth is governed by the Laplace equation [2,5,10]

$$\Delta u = 0 \ , \tag{3}$$

where $u = u(x,y)$ is the antiplane displacement field.

We break a center bond as the initial crack. We shall assume that crack growth is slow enough to allow the stresses to relax in such a way that the material is always at equilibrium.

The free energy of the system in the continuum limit is [11]

$$F = \frac{\lambda}{2} \int \int |\nabla u|^2 dx\, dy \ , \tag{4}$$

where λ is the Lamé coefficient. From Eqs. (3) and (4) we find that this antiplane problem is equivalent to a dielectric breakdown, though some differences between these two models exist [2,5]. Therefore, in the following we shall make much use of DB models.

We define the energy release rate G_i and the growth probability P_i for each growth bond i. The diffusion-limited-aggregation (DLA) growth law [12,13] leads to

$$P_i \propto \nabla_i \ , \tag{5}$$

where $\nabla_i = |(\mathbf{n}\cdot\nabla u)_i|$.

Combining the Griffith's condition and the DLA growth law, we have

$$P_i = \begin{cases} 0, & G_i < G_c \\ \dfrac{\nabla_i}{\sum\limits_{G_i \geq G_c} \nabla_i}, & G_i \geq G_c \end{cases} \ . \tag{6}$$

Halsey [14] has derived an identity for the change in the total energy of dielectric breakdown if the surface of the cluster moves by a small amount. This is

$$\delta\varepsilon = \frac{1}{8\pi} \int_s dw\, f(w)[P(w)]^2 \ , \tag{7}$$

where $-P(w) = \mathbf{E}(w)\cdot\mathbf{n}(w)$ is the normal electric field at w on the surface of the cluster, and $f(w)$ is the normal change in the surface position at w.

In two dimensions if a particle lands at w', the change in the surface is

$$f_{w'}(w) = ka^2\delta(w - w') \ , \tag{8}$$

where k is a numerical constant and a is the size of the individual particles in the cluster.

Substituting Eq. (8) into Eq. (7) yields

$$\begin{aligned} \delta\varepsilon &= \frac{1}{8\pi} ka^2[P(w')]^2 \\ &= \frac{1}{8\pi} ka^2[\mathbf{E}(w')\cdot\mathbf{n}(w')]^2 \ . \end{aligned} \tag{9}$$

Applying Eq. (9) to the antiplane crack-growth problem, we have

$$G_i \propto \delta\varepsilon_i \propto \nabla_i^2 , \quad (10)$$

where $\nabla_i = |(\mathbf{n}\cdot\nabla u)_i|$.

Using Eq. (10), Eq. (6) can be written as

$$P_i = \begin{cases} 0, & \nabla_i < \nabla_c \\ \dfrac{\nabla_i}{\sum\limits_{\nabla_i \ge \nabla_c} \nabla_i}, & \nabla_i \ge \nabla_c \end{cases} . \quad (11)$$

Thus from Eq. (11) we find that the model is equivalent to the dielectric breakdown model with cutoffs on the gradients, introduced by Arian *et al.* [8]. From their theoretical analysis and simulations, we deduce that the crack will evolve from DLA-type patterns (at small sizes) into spiky structures (at large sizes).

We next consider general crack growth, which is governed by the Lamé equation [11]

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) = 0 , \quad (12)$$

where $\mathbf{u}$ is the vectorial displacement field, λ and μ are the Lamé coefficients.

The DLA growth law [12,13] leads to a growth speed v_n that is proportional to the tangential tensions τ at the traction-free-crack surfaces [4], i.e.,

$$v_n \propto \tau . \quad (13)$$

Combining the Griffiths' condition and Eq. (13), we have

$$v_n \propto \begin{cases} 0, & G < G_c \\ \tau, & G \ge G_c \end{cases} . \quad (14)$$

We further consider a two-dimensional elastic medium under a fixed-displacement boundary condition, which is described as a triangular lattice with central forces between nearest-neighbors, corresponding to $\lambda = \mu$ in the continuum limit [4,6,7]. The growth probability of a growth bond is a function of the stress T stored in that bond, i.e., $P(T) = f(T)$. The DLA growth law leads to

$$P(T) \propto T . \quad (15)$$

From (14)

$$P_i = \begin{cases} 0, & T_i < T_c \\ \dfrac{T_i}{\sum\limits_{T_i \ge T_c} T_i}, & T_i \ge T_c \end{cases} . \quad (16)$$

We can use multifractal theory to analyze this problem [8,15]. There are $R^{f(\alpha)}d\alpha$ growth bonds, each with growth probability $R^{-\alpha}$, giving

$$\int R^{f(\alpha)-\alpha} d\alpha = 1 . \quad (17)$$

Applying the method of steepest descents to Eq. (17), it follows that the great majority of the growth takes place on the growth bonds with $\alpha_0 = f(\alpha_0) = D(q=1)$. Therefore, if

$$R^{-\alpha_0} > \frac{T_c}{\sum\limits_{T_i \ge T_c} T_i} , \quad (18)$$

the growth is not affected by T_c. So in this case the growth patterns resemble closely DLA clusters, with $D = 1.51$ and $D(q=1) \approx 1$ [6]. Since $\sum_{T_i \ge T_c} T_i \propto R^{\alpha_{\min}}$ [8], $\alpha_{\min} = D - 1$ [16], Eq. (18) can be written as

$$R < R_c , \quad (19)$$

where $R_c \propto T_c^{1/(D-\alpha_0-1)}$. When $R > R_c$, the growth will take place on the growth bonds with $P_{\max}$, which means that the growth will take place at the tips and the cluster will evolve into a spiky structure.

The growth patterns of the model resemble closely that of the crack-growth model introduced by Pla *et al.* [7], where the growth probability is given by

$$P(T) \propto T[1 + (T/T_0)^{\eta-1}] \quad (\eta \gg 1) .$$

In conclusion, crack growth governed by the Griffith condition and the DLA growth law will evolve from DLA-type patterns (at small sizes) into spiky structures (at large sizes), which can be used to explain the appearance of long and stringy cracks in many fracture processes.

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